Problem 30

In each of Problems 30 through 35, use the result of Problem 29 to find the Laplace transform of the given function; a and b are real numbers and n is a positive integer.

$$f(t) = te^{at}$$

Solution

The Laplace transform of a function $f(t)$ is defined here as

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt.$$

Substitute the given function and evaluate the integral.

$$
\begin{aligned}
F(s) &= \int_0^\infty e^{-st} t e^{at}\, dt \\
&= \int_0^\infty \left(-\frac{\partial}{\partial s} e^{-st}\right) e^{at}\, dt \\
&= -\frac{d}{ds} \int_0^\infty e^{-st} e^{at}\, dt \\
&= -\frac{d}{ds} \int_0^\infty e^{(a-s)t}\, dt \\
&= -\frac{d}{ds} \left[\frac{1}{a-s} e^{(a-s)t}\Big|_0^\infty\right] \\
&= -\frac{d}{ds} \left(\frac{1}{s-a}\right) \\
&= -\left[-\frac{1}{(s-a)^2}\right] \\
&= \frac{1}{(s-a)^2}
\end{aligned}
$$